ELSEVIER

Reduction of generalized complex structures

Mathieu Stiénon[*],[1], Ping Xu

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, United States

Abstract

We study reduction of generalized complex structures. More precisely, we investigate the following question. Let J be a generalized complex structure on a manifold M, which admits an action of a Lie group G preserving J. Assume that M_0 is a G-invariant smooth submanifold and the G-action on M_0 is proper and free so that $M_G := M_0/G$ is a smooth manifold. Under what condition does J descend to a generalized complex structure on M_G? We describe a sufficient condition for the reduction to hold, which includes the Marsden–Weinstein reduction of symplectic manifolds and the reduction of the complex structures in Kähler manifolds as special cases. As an application, we study reduction of generalized Kähler manifolds.
© 2007 Elsevier B.V. All rights reserved.

1. Introduction

Generalized complex structures [13,10] have been extensively studied recently due to their close connection with mirror symmetry. They include both symplectic and complex structures as extreme cases.

As is well known in symplectic geometry, a useful way of producing new symplectic manifolds is via the so called Marsden–Weinstein reduction [23]. This is a procedure for constructing symplectic manifolds from a Hamiltonian system with symmetry admitting a momentum map. Let us recall this construction briefly below. Suppose we are given a symplectic manifold (M, ω), an action of a Lie group G on M preserving the symplectic form, and an equivariant momentum map $\mu : M \to \mathfrak{g}^*$, i.e. μ satisfies the relations

$$\mu(g \cdot x) = \mathrm{Ad}^*_{g^{-1}} \mu(x), \quad \forall g \in G, x \in M$$

and

$$\hat{A} \lrcorner \omega = d\langle \mu, A \rangle, \quad \forall A \in \mathfrak{g},$$

where $\hat{A}$ denotes the fundamental vector field generated by $A \in \mathfrak{g}$. Assume that 0 is a regular value of the momentum map μ so that the preimage $M_0 = \mu^{-1}(0)$ is a G-invariant smooth submanifold. Moreover, if we assume that the

---

* Corresponding author.
 E-mail addresses: stienon@math.psu.edu (M. Stiénon), ping@math.psu.edu (P. Xu).

[1] Francqui fellow of the Belgian–American Educational Foundation.

G-action on M_0 is free and proper so that $M_G = M_0/G$ is a smooth manifold, then it inherits a natural symplectic structure [23] (see [27] for the symplectic reduction in the singular case). In the context of Poisson manifolds, a reduction procedure was carried out by Marsden and Ratiu [24] and Ortega and Ratiu for the singular case [25]. See [26] for a beautiful comprehensive study on Hamiltonian reductions.

On the other hand, since there is no such notion of momentum maps for complex manifolds, there does not exist a general scheme of reduction of G-invariant complex structures in the literature as far as we know. However, for a G-invariant Kähler manifold which admits an equivariant momentum map for the symplectic structure, one can prove that the complex structure can also descend to the symplectic reduced space $M_G = M_0/G$ so that M_G becomes a Kähler manifold. This is the so called Kähler reduction.

There are several versions of Kähler reduction, which were due to Guillemin and Sternberg [11], Kirwan [16], and Hitchin et al. [12] respectively. A careful examination of the argument used by Guillemin and Sternberg in [11] shows that the following identity:

$$T_x M = T_x M_0 \oplus j(T_x(G \cdot x)), \quad \forall x \in M_0, \tag{1}$$

plays an essential role in carrying out the reduction for the complex structure, where $j : TM \to TM$ is the complex structure of the Kähler manifold. Eq. (1) holds automatically when $M_0 = \mu^{-1}(0)$ is the level set of the momentum map $\mu : M \to \mathfrak{g}^*$. Indeed this is exactly why the symplectic reduced space $M_G = M_0/G$ inherits a complex structure.

A natural question arises as to whether there is a reduction procedure for a G-invariant generalized complex structure which combines the above two special cases. More precisely, the question can be formulated as follows.

> Let J be a generalized complex structure on a manifold M, which admits an action of a Lie group G preserving J. Assume that M_0 is a G-invariant smooth submanifold and the G-action on M_0 is free and proper so that $M_G := M_0/G$ is a smooth manifold. Under what condition does J descend to a generalized complex structure on M_G?

In this paper, we describe a sufficient condition for such a reduction. Our condition comprises many well-known examples as special cases including the above two important cases. In particular, even when J is an honest complex structure, we derive some interesting condition for the reduction of the complex structure to hold, which seems to be new.

Below let us describe the main idea of our approach briefly. There are several equivalent definitions of generalized complex structures. A useful one for us is that a generalized complex structure on a manifold M is a pair of transversal (complex) Dirac structures (E_+, E_-) on M which are complex conjugate to each other. In other words (E_+, E_-) constitutes a (complex) Lie bialgebroid in the sense of [20] (see [19]). Roughly, our approach is as follows. Using the inclusion map $M_0 \to M$, one may pull back $E_\pm$ to M_0 to obtain a pair of complex conjugate Dirac structures on M_0. Since J is G-invariant, using the push forward map $M_0 \to M_0/G = M_G$, one obtains a pair of complex conjugate Dirac structures $E'_\pm$ on M_G. To ensure that this gives rise to a generalized complex structure on M_G, a necessary condition is that $E'_+ \cap E'_- = 0$. However, there is a subtlety. It is due to the difficulty that the pull back and push forward of Dirac structures may not be smooth bundles. This forces us to impose some extra smoothness assumptions. Indeed, in this paper we combine the pull back and push forward steps together to derive a sufficient condition for the reduction. Note that Dirac reduction in the real context was first studied by Blankenstein (see [3,4]) and by Blankenstein and Ratiu for the singular case [5] in connection with the study of Hamiltonian mechanics.

One may easily check that if one starts with a symplectic structure, the reduced generalized complex structure is still symplectic, while reduction of a complex structure is still a complex structure. Thus the two examples above are indeed special cases of our general condition. As an application, we study reduction of generalized Kähler manifolds, which generalizes the usual Kähler reduction [11,12,16]. Our paper only deals with the analogue of regular Kähler reduction. See [15] for the singular case.

The paper is organized as follows. In Section 2, we recall some basic materials in Dirac geometry. In particular, we describe the pull back and push forward constructions. In Section 3, we recall several equivalent definitions of generalized complex structures. Section 4 is devoted to the study of reduction of generalized complex structures. As corollaries, we consider several special examples including the reduction of usual complex structures, and the B-transform of symplectic structures. In Section 5, we give an explicit description of the reduced generalized complex structure in terms of the endomorphism J_G of the vector bundle $TM_G \oplus T^*M_G$. In Section 6, as an application, we investigate the reduction of generalized Kähler structures.

As a special case when $G = \{*\}$, our main results would lead to conditions which guarantee submanifolds of a generalized complex manifold inheriting natural generalized complex structures. This topic was studied in detail separately by Barton and the first author [1]. Note though that the first investigation of the submanifolds of a generalized complex manifold was [2].

The results of this paper were announced at the conference "Poisson geometry" held in Trieste in July 2005, where we learned that Bursztyn, Cavalcanti and Gualtieri [6] were working on a related subject. Subsequently, we noticed that there have appeared several papers studying similar topics independently, including the one by Hu [14], the one by Lin and Tolman [21] and the one by Vaisman [29]. It would be interesting to clarify the relations between all these approaches.

Notation. We denote by $V_{\mathbb{C}}$ the complexified $V \otimes \mathbb{C}$ of a vector space V. Likewise, $T_{\mathbb{C}}M$ and $T_{\mathbb{C}}^*M$ respectively denote the tangent and cotangent bundles of a manifold M. If W is a vector subspace of V, W^0 denotes the annihilator subspace of W inside V^*. And if V is endowed with an inner product, $W^{\perp}$ denotes the subspace of V orthogonal to W. Finally, $\mathbf{i}$ denotes the imaginary number $\sqrt{-1}$.

2. Dirac geometry

In this section, we briefly review some basic ingredients of Dirac geometry which will be used in this paper. For details, see [7,8].

Given a vector space V, we consider the direct sum $V \oplus V^*$ endowed with the *inner product*

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\left(\xi(Y) + \eta(X) \right), \quad \forall X, Y \in V, \xi, \eta \in V^*.$$

We denote by ρ and ρ_* the natural projections of $V \oplus V^*$ onto V and V^* respectively. A *Dirac structure* on V is a maximal isotropic subspace of $V \oplus V^*$ with respect to $\langle \cdot, \cdot \rangle$. The set of linear Dirac structures on V is denoted by $\mathrm{Dir}(V)$.

Alternatively, a Dirac structure on a vector space can be described as follows. Let $L \in \mathrm{Dir}(V)$. Since L is isotropic with respect to $\langle \cdot, \cdot \rangle$, there exists a natural skew-symmetric bilinear form Λ on L defined by

$$\Lambda(X + \xi, Y + \eta) = \xi(Y) = -\eta(X), \quad \forall X + \xi, Y + \eta \in L.$$

One easily checks that

$$\Lambda(X + \xi_1, Y + \eta_1) = \Lambda(X + \xi_2, Y + \eta_2), \quad \forall X + \xi_i, Y + \eta_i \in L, \ i = 1, 2$$

and

$$\Lambda(X_1 + \xi, Y_1 + \eta) = \Lambda(X_2 + \xi, Y_2 + \eta), \quad \forall X_i + \xi, Y_i + \eta \in L, \ i = 1, 2.$$

Hence, there exists a 2-form Ω on $\rho(L)$ and a 2-form π on $\rho_*(L)$ defined, respectively, by

$$\Omega(X, Y) = \Lambda(X + \xi, Y + \eta) = -\pi(\xi, \eta), \quad \forall X + \xi, Y + \eta \in L.$$

Let $X \in \rho(L)$. Then there is a $\xi \in V^*$ such that $X + \xi \in L$. And we have $\Omega(X, Y) = \xi(Y), \forall Y \in \rho(L)$, or, equivalently, $X \lrcorner \Omega = \xi_{|\rho(L)}$, where $\xi_{|\rho(L)}$ denotes the restriction of $\xi \in V^*$ to $\rho(L)$. Thus

$$X + \xi \in L \iff \begin{cases} X \in \rho(L) \\ X \lrcorner \Omega = \xi_{|\rho(L)}. \end{cases}$$

Thus knowing the Dirac structure L on V is exactly the same as knowing the subspace $\rho(L)$ of V and the 2-form Ω on $\rho(L)$. Similarly, L is equivalent to the pair $(\rho_*(L), \pi)$.

Therefore, we have the converse: any subspace $R \subset V$ endowed with a 2-form Ω defines a Dirac structure

$$\{X + \xi \mid X \in R, \xi \in V^*, X \lrcorner \Omega = \xi_{|R}\}$$

on V and any subspace $R^* \subset V^*$ endowed with a 2-form π defines a Dirac structure

$$\{X + \xi \mid \xi \in R^*, X \in V, \pi(\xi, \eta) = -\eta(X), \forall \eta \in R^*\}$$

on V.

Dirac structures can be pulled back and pushed forward using a linear map. Let $V \xrightarrow{\phi} W$ be a linear map. Consider a Dirac structure L_W on W and its associated pair $(\rho(L_W), \Omega_{L_W})$. The 2-form Ω_{L_W} pulls back to the preimage $\phi^{-1}(\rho(L_W))$, yielding the pair $(\phi^{-1}(\rho(L_W)), \phi^*\Omega_{L_W})$. The Dirac structure on V associated with this new pair is called the *backward* image of L_W through ϕ and is denoted by $\mathcal{B}\phi(L_W)$.

Let $W^* \xrightarrow{\phi^*} V^*$ be the dual of ϕ. Consider a Dirac structure L_V on V and its associated pair $(\rho_*(L_V), \pi_{L_V})$. The pullback of the 2-form π_{L_V} to the preimage $(\phi^*)^{-1}(\rho_*(L_V))$ is equal to $\phi\pi_{L_V}$. The Dirac structure on W associated with the pair $((\phi^*)^{-1}(\rho_*(L_V)), \phi\pi_{L_V})$ is called the *forward* image of L_V through ϕ and is denoted by $\mathcal{F}\phi(L_V)$.

One easily checks that

$$\mathcal{B}\phi(L_W) = \{X + \phi^*\xi \,|\, X \in V, \xi \in W^*, \phi X + \xi \in L_W\}$$

and

$$\mathcal{F}\phi(L_V) = \{\phi(X) + \xi \,|\, X \in V, \xi \in W^*, X + \phi^*\xi \in L_V\}.$$

Let M be a smooth manifold. A *Dirac structure* on M is a smooth subbundle $L \subset TM \oplus T^*M$ which determines a Dirac structure in $T_mM \oplus T_m^*M$ for each $m \in M$ and whose space of sections is closed under the (skew-symmetric) *Courant bracket* on $\Gamma(TM \oplus T^*M)$ defined by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}d(\xi(Y) - \eta(X)), \quad \forall X, Y \in \mathfrak{X}(M), \ \xi, \eta \in \Omega^1(M).$$

In general, the Courant bracket does not satisfy the Jacobi identity. However, it does when restricted to the space $\Gamma(L)$ of sections of a Dirac subbundle $L \subset TM \oplus T^*M$. Then $L \to M$ inherits a Lie algebroid structure with the projection $L \xrightarrow{\rho|_L} TM$ as anchor map and the restriction of the Courant bracket as Lie algebroid bracket [8].

Although the backward and forward images of a Dirac structure through the differential of a smooth map are always pointwise Dirac structures, they are generally not constant rank smooth vector bundles. The remainder of this section is devoted to the description of situations in which backward and forward images of Dirac structures are again Dirac structures.

Definition 2.1. Let $M \xrightarrow{\varphi} N$ be a smooth map. Two generalized tangent vectors $X + \xi \in TM \oplus T^*M$ and $Y + \eta \in TN \oplus T^*N$ are said to be φ-*related*, denoted by $X + \xi \xrightarrow{\varphi} Y + \eta$, if $\varphi_*X = Y$ and $\varphi^*\eta = \xi$. Clearly, this notion extends to sections.

Lemma 2.2. *Assume that $\sigma_M^i \in \Gamma(TM \oplus T^*M)$ and $\sigma_N^i \in \Gamma(TN \oplus T^*N)$ satisfy $\sigma_M^i \xrightarrow{\varphi} \sigma_N^i$, where $i = 1, 2$. Then*

$$[\sigma_M^1, \sigma_M^2] \xrightarrow{\varphi} [\sigma_N^1, \sigma_N^2],$$

where the bracket on the LHS refers to the Courant bracket on $\Gamma(TM \oplus T^*M)$ while the bracket on the RHS refers to the Courant bracket on $\Gamma(TN \oplus T^*N)$.

Proof. Write $\sigma_M^i = X^i + \xi^i$ and $\sigma_N^i = Y^i + \eta^i$, where $X^i \in \mathfrak{X}(M)$, $\xi^i \in \Omega^1(M)$, $Y^i \in \mathfrak{X}(N)$ and $\eta^i \in \Omega^1(N)$ for $i = 1, 2$. Since σ_M^i and σ_N^i are φ-related, $\varphi_*X^i = Y^i$ and $\varphi^*\eta^i = \xi^i$ for $i = 1, 2$. Then

$$[\sigma_M^1, \sigma_M^2] = [X^1, X^2] + \mathcal{L}_{X^1}\xi^2 - \mathcal{L}_{X^2}\xi^1 + \frac{1}{2}d\left(\xi^1(X^2) - \xi^2(X^1)\right)$$

and

$$[\sigma_N^1, \sigma_N^2] = [Y^1, Y^2] + \mathcal{L}_{Y^1}\eta^2 - \mathcal{L}_{Y^2}\eta^1 + \frac{1}{2}d\left(\eta^1(Y^2) - \eta^2(Y^1)\right).$$

Now

$$\varphi_*[X^1, X^2] = [\varphi_*X^1, \varphi_*X^2] = [Y^1, Y^2],$$

and

$$
\begin{aligned}
\varphi^*(\mathcal{L}_{Y^1}\eta^2) &= \varphi^*(i_{Y^1}d\eta^2 + di_{Y^1}\eta^2) \\
&= \varphi^* i_{\varphi_* X^1}d\eta^2 + \varphi^* d\left(\eta^2(Y^1)\right) \\
&= i_{X^1}\varphi^* d\eta^2 + d\left(\xi^2(X^1)\right) \\
&= i_{X^1}d\xi^2 + di_{X^1}\xi^2 \\
&= \mathcal{L}_{X^1}\xi^2.
\end{aligned}
$$

Similarly, we have

$$
\varphi^*(\mathcal{L}_{Y^2}\eta^1) = \mathcal{L}_{X^2}\xi^1.
$$

Finally

$$
\varphi^* d\left(\eta^1(Y^2) - \eta^2(Y^1)\right) = d\left(\xi^1(X^2) - \xi^2(X^1)\right)
$$

since

$$
\varphi^*\left(\eta^1(Y^2)\right) = \varphi^*\left(\eta^1(\varphi_* X^2)\right) = (\varphi^*\eta^1)(X^2) = \xi^1(X^2).
$$

This concludes the proof. $\quad\square$

Recall that a *generalized smooth subbundle* of a vector bundle $E \to M$ is a subbundle $V \subset E$ (whose fibers may not be of constant ranks) such that for any vector $v \in V_m$ there is a local smooth section s of V extending v, i.e. $s|_m = v$.

Proposition 2.3. (1) *Let $M \xrightarrow{\ i\ } N$ be an embedding and L_N a Dirac structure on N. If, for any vector $v_m \in \mathcal{B}i(L_N)$, there exists a local smooth section s of L_N defined in a neighborhood of $i(m) \in N$, such that $s_{|M} \in \Gamma_{\text{loc}}(i_* TM \oplus T^*N|_M)$ and $v_m \overset{i}{\rightsquigarrow} s|_{i(m)}$, then $\mathcal{B}\phi(L_N)$ is a Dirac structure on M. In particular, the above condition holds if $L_N|_M \cap (i_*(TM) \oplus T^*N|_M)$ is a generalized smooth subbundle.*

(2) *Let M be a smooth manifold endowed with a proper and free action of a Lie group G. Let π denote the canonical projection of M onto $N := M/G$. If L_M is a G-invariant Dirac structure on M, and for any vector $v_n \in \mathcal{F}\pi(L_M)$, there exists a local smooth section s of L_M defined in a neighborhood of a point $m \in M$ in the fiber of π over n such that $s \in \Gamma_{\text{loc}}(TM \oplus \pi^*(T^*N))$ and $s|_m \overset{\pi}{\rightsquigarrow} v_n$, for instance when $L_M \cap (TM \oplus \pi^*(T^*N))$ is a generalized smooth subbundle, then $\mathcal{F}\pi(L_M)$ is a Dirac structure on N.*

Proof. (1). First, we observe that $\mathcal{B}i(L_N)$ is well defined since i is injective. The assumption implies that any vector $v_m \in \mathcal{B}i(L_N)$ admits a local section $\tilde{s}$ of $\mathcal{B}i(L_N)$ such that $\tilde{s} \overset{i}{\rightsquigarrow} s$, where s is a local smooth section of L_N defined in a neighborhood of $i(m) \in N$ such that $s_{|M} \in \Gamma_{\text{loc}}(i_* TM \oplus T^*N|_M)$. Then $\tilde{s}$ must be a smooth local section. Therefore $\mathcal{B}i(L_N)$ is a smooth subbundle.

Now for any point $m \in M$, take a basis $\{v^k \in \mathcal{B}i(L_N)|_m\}_{k \in K}$. Let $\tilde{s}^k$ be a local section of $\mathcal{B}i(L_N)$ through v^k as above. Then $\tilde{s}^k \overset{i}{\rightsquigarrow} s^k$, where s^k is a local smooth section of L_N. Thus $[\tilde{s}^k, \tilde{s}^l]$, $\forall k, l$, is still a local smooth section of $\mathcal{B}i(L_N)$ according to Lemma 2.2. Hence it follows that $\mathcal{B}i(L_N)$ is Courant involutive.

For the second part, note that any element $v_m \in \mathcal{B}i(L_N)$ naturally induces an element $v_{i(m)} \in L_N|_M \cap (i_*(TM) \oplus T^*N|_M)$ such that $v_m \overset{i}{\rightsquigarrow} v_{i(m)}$. Thus the condition is satisfied since $L_N|_M \cap (i_*(TM) \oplus T^*N|_M)$ is a generalized smooth subbundle by assumption.

(2). First, we observe that $\mathcal{F}\pi(L_M)$ is well defined since L_M is G-invariant. Now for any vector $v_n \in \mathcal{F}\pi(L_M)$, let s be a local section of L_M defined in a neighborhood of a point $m \in \pi^{-1}(n)$ such that $s \in \Gamma_{\text{loc}}(TM \oplus \pi^*(T^*N))$ and $s|_m \overset{\pi}{\rightsquigarrow} v_n$. Let $L \subset M$ be a slice through m transversal to the G-orbits. Since L_M is G-invariant, there is a local smooth section $\hat{s}$ of L_M such that $\hat{s}|_L = s|_L$. It is clear that $\hat{s} \in \Gamma_{\text{loc}}(TM \oplus \pi^*(T^*N))$. Hence it induces a local section $\tilde{s}$ of $\mathcal{F}\pi(L_M)$ through v_n. Therefore $\mathcal{F}\pi(L_M)$ is indeed a smooth subbundle. The rest can be proved like in (1). $\quad\square$

3. Generalized complex structures

Let V be a vector space. Consider a linear endomorphism J of $V \oplus V^*$ such that $J^2 = -I$ and J is orthogonal with respect to the inner product

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} \left(\xi(Y) + \eta(X) \right), \quad \forall X, Y \in V, \xi, \eta \in V^*.$$

Such a linear map is called a *linear generalized complex structure* by Hitchin and Gualtieri [10,13]. The complexified vector space $(V \oplus V^*) \otimes \mathbb{C}$ decomposes as the direct sum

$$(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$$

of the eigenbundles of J corresponding to the eigenvalues $\pm \mathbf{i}$ respectively, i.e.,

$$E_\pm = \{ (X + \xi) \mp \mathbf{i} J (X + \xi) | X + \xi \in V \oplus V^* \}.$$

Both eigenspaces are maximal isotropic with respect to $\langle \cdot, \cdot \rangle$ and they are complex conjugate to each other.

The following lemma is obvious.

Lemma 3.1. *The linear generalized complex structures are in 1–1 correspondence with the splittings $(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$ with $E_\pm$ maximal isotropic and $E_- = \overline{E_+}$.*

Now, let M be a manifold and J a bundle endomorphism of $TM \oplus T^*M$ such that $J^2 = -I$, and J is orthogonal with respect to $\langle \cdot, \cdot \rangle$. In the associated eigenbundle decomposition

$$(T_\mathbb{C} M \oplus T_\mathbb{C}^* M) = E_+ \oplus E_-,$$

if $\Gamma(E_+)$ is closed under the (complexified) Courant bracket, then E_+ is a (complex) Dirac structure on M and one says that J is a *generalized complex structure* [10,13]. In this case, E_- must also be a Dirac structure since $E_- = \overline{E_+}$. Indeed (E_+, E_-) is a complex Lie bialgebroid in the sense of [20], in which E_+ and E_- are complex conjugate to each other.

The following proposition gives two equivalent definitions of a generalized complex structure.

Proposition 3.2. *A generalized complex structure is equivalent to any of the following:*

(1) *A bundle endomorphism J of $TM \oplus T^*M$ such that J is orthogonal with respect to $\langle \cdot, \cdot \rangle$ and $J^2 = -I$ satisfying*

$$[Je_1, Je_2] - [e_1, e_2] - J([Je_1, e_2] + [e_1, Je_2]) = 0, \quad \forall e_1, e_2 \in \Gamma(TM \oplus T^*M). \tag{2}$$

(2) *A complex Lie bialgebroid (E_+, E_-) such that its double is the standard Courant algebroid $T_\mathbb{C} M \oplus T_\mathbb{C}^* M$, and E_+ and E_- are complex conjugate to each other.*

Proof. (1) $\Longrightarrow$ (2) As above, $E_\pm$ are the $(\pm \mathbf{i})$-eigenbundles of J. Since Eq. (2) exactly means that

$$[e_1 \mp \mathbf{i} J e_1, e_2 \mp \mathbf{i} J e_2] \in \Gamma(E_\pm), \quad \forall e_1, e_2 \in \Gamma(TM \oplus T^*M),$$

it is equivalent to the involutivity of $\Gamma(E_\pm)$.

(2) $\Longrightarrow$ (1) Define $J : T_\mathbb{C} M \oplus T_\mathbb{C}^* M \to T_\mathbb{C} M \oplus T_\mathbb{C}^* M$ by setting $J|_{E_+} = \mathbf{i} I$ and $J|_{E_-} = -\mathbf{i} I$. Since E_+ and E_- are complex conjugate to each other, we have $J \bar{e} = \overline{Je}, \forall e \in T_\mathbb{C} M \oplus T_\mathbb{C}^* M$. Therefore J must be the $\mathbb{C}$-linear extension of an endomorphism of $TM \oplus T^*M$. It is easy to check that J does indeed satisfy all the axioms as in (1). $\square$

The following are two standard examples [10,13].

Examples 3.3. (1) Let j be an almost complex structure on M. Then

$$J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$$

is $\langle \cdot, \cdot \rangle$-orthogonal and satisfies $J^2 = -I$. J is a generalized complex structure if and only if j is integrable.

(2) Let ω be a non-degenerate 2-form on M. Then

$$J = \begin{pmatrix} 0 & -\omega_\flat^{-1} \\ \omega_\flat & 0 \end{pmatrix},$$

where $\omega_\flat : TM \to T^*M$ is the bundle map $X \mapsto X \lrcorner \omega$, is a generalized complex structure if and only if $d\omega = 0$, i.e., ω is a symplectic 2-form.

4. Reduction using Dirac structures

Let M be a manifold, J a generalized complex structure on M and φ an action of a Lie group G on M preserving the generalized complex structure J. In other words, we have a group homomorphism $\varphi : G \to \text{Diffeo}(M) : g \mapsto \varphi_g$ such that, $\forall g \in G$, the bundle endomorphism

$$
\begin{array}{ccc}
TM \oplus T^*M & \xrightarrow{\phi_g} & TM \oplus T^*M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\varphi_g} & M
\end{array}
$$

(defining the induced G-action ϕ on $TM \oplus T^*M$) commutes with J, where $\phi_g = \varphi_{g*} \oplus \varphi_{g^{-1}}^*$. As in Section 3, by $E_\pm$ we denote the ($\pm\mathbf{i}$)-eigenbundles of J.

Let M_0 be a G-invariant submanifold of M with a free and proper G-action so that the quotient space $M_G := M_0/G$ is a smooth manifold. Thus one has the following maps:

$$M \xleftarrow{i} M_0 \xrightarrow{\pi} M_G.$$

The main question that we investigate in this section is: under what condition does M_G inherit a generalized complex structure?

Since π is surjective and J is G-invariant, $E'_\pm := \mathcal{F}\pi \, (\mathcal{B}i(E_\pm))$ are well-defined, complex conjugate pointwise Dirac structures on M_G. It is simple to check that

$$E'_\pm = \{\pi_* X + \xi' | X \in T_\mathbb{C} M_0, \xi' \in T^*_\mathbb{C} M_G \text{ and } \exists \xi \in T^*_\mathbb{C} M \text{ st } i_* X + \xi \in E_\pm \text{ and } \pi^* \xi' = i^* \xi\}.$$

One possible way to ensure that $E'_\pm$ are Dirac structures is to use Proposition 2.3 by considering the pull back Dirac structure $\mathcal{B}i(E_\pm)$ and then the push forward structure $\mathcal{F}\pi \, (\mathcal{B}i(E_\pm))$. Below, however, we will combine these two steps together, which enables us to obtain a stronger result.

In what follows, the foliation of M_0 defined by the G-orbits will be denoted by $\mathcal{F}$, and $(T\mathcal{F})^0$ and $(TM_0)^0$ will denote the annihilators of $T\mathcal{F}$ and TM_0 in TM along the submanifold M_0 respectively.

It is clear that any element $i_* X_m + \xi_{i(m)} \in TM_0 \oplus (T\mathcal{F})^0$ induces an element $\pi_* X_m + \xi'_{\pi(m)} \in TM_G \oplus T^*M_G$ where $\xi'_{\pi(m)}$ is defined by $i^* \xi_{\pi(m)} = \pi^* \xi'_{\pi(m)}$. Indeed, one has a bundle map

$$
\begin{array}{ccc}
TM_0 \oplus (T\mathcal{F})^0 & \xrightarrow{\Pi} & TM_G \oplus T^*M_G \, . \\
\downarrow & & \downarrow \\
M_0 & \xrightarrow{\pi} & M_G
\end{array}
\tag{3}
$$

The element $i_* X_m + \xi_{i(m)} \in TM_0 \oplus (T\mathcal{F})^0$ is called a *lift* of $\pi_* X_m + \xi'_{\pi(m)} \in TM_G \oplus T^*M_G$. Note that, for a given element $v_n \in TM_G \oplus T^*M_G$, there exist many different lifts. Indeed, there is a choice of point m in the fiber $\pi^{-1}(n) \subset M_0$ involved and Π has $T\mathcal{F} \oplus (TM_0)^0$ as non-trivial kernel.

Proposition 4.1. *The following assertions are equivalent.*

(1) $E'_+ \cap E'_- = 0$.
(2) $(E_+ + T_\mathbb{C}\mathcal{F}) \cap (T_\mathbb{C} M_0 \oplus (T_\mathbb{C}\mathcal{F})^0) \cap (E_- + (T_\mathbb{C} M_0)^0) \subset T_\mathbb{C}\mathcal{F} \oplus (T_\mathbb{C} M_0)^0$.

(3) *If* $z \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$ *with* $z = z_+ + z_-$ *such that* $z_\pm \in E_\pm$ *and* $z_\pm \in T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0$, *then* $z_\pm \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$.

(4) $J\left(T\mathcal{F} \oplus (TM_0)^0\right) \cap \left(TM_0 \oplus (T\mathcal{F})^0\right) \subset T\mathcal{F} \oplus (TM_0)^0$.

(5) $J(TM_0 \oplus (T\mathcal{F})^0) \subset (TM_0 \oplus (T\mathcal{F})^0) + J(T\mathcal{F} \oplus (TM_0)^0)$.

(6) $TM_0 \oplus (T\mathcal{F})^0 = (TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0) + (T\mathcal{F} \oplus (TM_0)^0)$.

(7) $T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G = E'_+ + E'_-$.

Proof. (1) $\Leftrightarrow$ (2) Note that

$$E'_+ \cap E'_- = \{X' + \xi' \in T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G | \exists X_\pm \in T_{\mathbb{C}}M_0 \text{ and } \xi_\pm \in T_{\mathbb{C}}^*M_{|M_0}$$
$$\text{such that } i_*X_\pm + \xi_\pm \in E_\pm, \pi_*X_\pm = X' \text{ and } i^*\xi_\pm = \pi^*\xi'\}.$$

Thus, in particular, we have

$$X_\pm \in T_{\mathbb{C}}M_0 \qquad \xi_\pm \in (T_{\mathbb{C}}\mathcal{F})^0$$
$$X_- - X_+ \in T_{\mathbb{C}}\mathcal{F} \qquad \xi_- - \xi_+ \in (T_{\mathbb{C}}M_0)^0.$$

Assume that there exists an element $X' + \xi' \neq 0$ in $E'_+ \cap E'_-$. Then there exist $i_*X_\pm + \xi_\pm \in E_\pm$ with $X_\pm \in T_{\mathbb{C}}M_0$ and $\xi_\pm \in T_{\mathbb{C}}^*M_{|M_0}$ as above. In particular,

$$i_*X_- + \xi_+ \in (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0).$$

Since $T_{\mathbb{C}}\mathcal{F} \subset T_{\mathbb{C}}M_0$, we get

$$i_*X_- + \xi_+ = (i_*X_+ + \xi_+) + i_*(X_- - X_+) \in E_+ + T_{\mathbb{C}}\mathcal{F}.$$

On the other hand, we have

$$i_*X_- + \xi_+ = (i_*X_- + \xi_-) + (\xi_+ - \xi_-) \in E_- + (T_{\mathbb{C}}M_0)^0.$$

Hence

$$i_*X_- + \xi_+ \in (E_+ + T_{\mathbb{C}}\mathcal{F}) \cap (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0) \cap \left(E_- + (T_{\mathbb{C}}M_0)^0\right).$$

Now $X' + \xi' \neq 0$ implies that $i_*X_- + \xi_+$ must not belong to $T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$. Thus, we have proved that $E'_+ \cap E'_- \neq 0$ implies

$$(E_+ + T_{\mathbb{C}}\mathcal{F}) \cap (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0) \cap \left(E_- + (T_{\mathbb{C}}M_0)^0\right) \not\subset T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0.$$

It is an easy matter to show that the converse is also true.

(2) $\Rightarrow$ (3) Take $\hat{A} + f \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$ and assume that $\hat{A} + f = z_+ + z_-$ with $z_\pm \in E_\pm$ and $z_\pm \in T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0$. Then $z_+ - \hat{A} = -z_- + f$ is an element of the triple intersection of the LHS of (2). Hence $z_\pm \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$.

(3) $\Rightarrow$ (2) Any element z in the triple intersection can be written both as $v + \hat{A}$ and as $w + f$ for some $v \in E_+$, $\hat{A} \in T_{\mathbb{C}}\mathcal{F}$, $w \in E_-$ and $f \in (T_{\mathbb{C}}M_0)^0$. We have $w - v = \hat{A} - f \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$. The assumption (3) implies that $v, w \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$. And, finally, $z = v + \hat{A} \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$.

(3) $\Leftrightarrow$ (4) Since $T_{\mathbb{C}}M = E_+ \oplus E_-$, the only solution to $z = z_+ + z_-$, where $z \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$ and $z_\pm \in E_\pm$, is $z_\pm = \frac{1}{2}(z \mp \mathbf{i}Jz)$. Since $T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0 \subset T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0$, one has $z_\pm \in T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0$ if and only if $Jz \in T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0$. Thus condition (3) can be rewritten as

$$\left.\begin{matrix} z \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0 \\ Jz \in T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0 \end{matrix}\right\} \Longrightarrow Jz \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$$

which is equivalent to (4).

(4) $\Leftrightarrow$ (5) Dualizing one condition by taking the annihilator with respect to $\langle \cdot, \cdot \rangle$ and applying J to both sides yields the other one.

$(5) \Leftrightarrow (6)$ Since $J^2 = -I$, (5) is equivalent to

$$TM_0 \oplus (T\mathcal{F})^0 \subset J(TM_0 \oplus (T\mathcal{F})^0) + (T\mathcal{F} \oplus (TM_0)^0).$$

Remembering that $T\mathcal{F} \oplus (TM_0)^0 \subset TM_0 \oplus (T\mathcal{F})^0$, the equivalence with (6) is obvious.

$(6) \Rightarrow (7)$ Any element $v \in T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G$ has a lift $\tilde{v} \in T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0$. By (6), there exists $k \in T_{\mathbb{C}}\mathcal{F} \oplus (T_{\mathbb{C}}M_0)^0$ such that $\tilde{v} - k \in (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0) \cap J(T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0)$. Then the vectors $\frac{1}{2}(I - \mathbf{i}J)(\tilde{v} - k)$ and $\frac{1}{2}(I + \mathbf{i}J)(\tilde{v} - k)$ belong to $E_+ \cap (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0)$ and $E_- \cap (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0)$ respectively and are thus the lifts of a pair of vectors $v_+ \in E'_+$ and $v_- \in E'_-$ such that $v = v_+ + v_-$.

$(7) \Rightarrow (6)$ Let $\hat{v}$ be a vector of $TM_0 \oplus (T\mathcal{F})^0$. It is a lift of some vector $v \in TM_G \oplus T^*M_G$. Hence v can be written as a sum $v = v_+ + v_-$, where $v_+ \in E'_+$ and $v_- \in E'_-$. Choose lifts $\tilde{v}_+ \in E_+ \cap (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0)$ and $\tilde{v}_- \in E_- \cap (T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0)$ of v_+ and v_- respectively. Then $\tilde{v}_+ + \tilde{v}_-$ is a lift of v and belongs to $(T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0) \cap J(T_{\mathbb{C}}M_0 \oplus (T_{\mathbb{C}}\mathcal{F})^0)$. And $\tilde{v} = \mathrm{Re}(\tilde{v}_+ + \tilde{v}_-)$ is still a lift of v since v is a real vector. Moreover $\tilde{v} \in (TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$. Let $k = \hat{v} - \tilde{v}$. Then it follows that $k \in T\mathcal{F} \oplus (TM_0)^0$ since both $\hat{v}$ and $\tilde{v}$ are lifts of v. $\square$

As an immediate consequence, we have

Corollary 4.2. *The direct sum decomposition $T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G = E'_+ \oplus E'_-$ holds if and only if any element of $TM_G \oplus T^*M_G$ has a lift in $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$.*

Proof. By Proposition 4.1, the direct sum decomposition holds if and only if

$$TM_0 \oplus (T\mathcal{F})^0 = (TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0) + (T\mathcal{F} \oplus (TM_0)^0).$$

The result thus follows from the fact that $\ker \Pi = T\mathcal{F} \oplus (TM_0)^0$ and the surjectivity of Π, where Π is the map as in Eq. (3). $\square$

It is clear that any G-invariant (local) section $\tilde{\sigma}$ of $TM_0 \oplus (T\mathcal{F})^0 \to M_0$ induces a (local) section σ of $TM_G \oplus T^*M_G \to M_G$. Here, G-invariant local sections must be understood in the following obvious sense: $\tilde{\sigma}(m_1) = \phi_g \cdot \tilde{\sigma}(m_2)$, for all $m_1, m_2 \in U$ such that $m_1 = g \cdot m_2$ for some $g \in G$. Hence, the bundle map (3) induces a map

$$\Gamma_{\mathrm{loc}}(TM_0 \oplus (T\mathcal{F})^0)^G \longrightarrow \Gamma_{\mathrm{loc}}(TM_G \oplus T^*M_G).$$

Definition 4.3. Let U be an open neighborhood in M. A local section $\tilde{\sigma} \in \Gamma((TM \oplus T^*M)_{|U})^G$ is said to be a *lift of a local section* $\sigma \in \Gamma((TM_G \oplus T^*M_G)_{|\pi(U \cap M_0)})$ if $\tilde{\sigma}_{|U \cap M_0} \in \Gamma((TM_0 \oplus (T\mathcal{F})^0)_{|U \cap M_0})$ and the diagram

$$
\begin{array}{ccc}
\left(TM_0 \oplus (T\mathcal{F})^0\right)_{|U \cap M_0} & \xrightarrow{\quad \Pi \quad} & (TM_G \oplus T^*M_G)_{|\pi(U \cap M_0)} \\[2mm]
{\scriptstyle \tilde{\sigma}_{|U \cap M_0}} \big\uparrow & & \big\uparrow {\scriptstyle \sigma} \\[2mm]
U \cap M_0 & \xrightarrow{\quad \pi \quad} & \pi(U \cap M_0)
\end{array}
\tag{4}
$$

commutes.

Lemma 4.4. *Given any local sections $\sigma_1, \sigma_2 \in \Gamma_{\mathrm{loc}}(TM_G \oplus T^*M_G)$, assume that $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \Gamma_{\mathrm{loc}}(TM \oplus T^*M)^G$ are any of their lifts. Then*

(1) $[\tilde{\sigma}_1, \tilde{\sigma}_2] \in \Gamma_{\mathrm{loc}}(TM \oplus T^*M)^G$,
(2) $[\tilde{\sigma}_1, \tilde{\sigma}_2]_{|M_0} \in \Gamma_{\mathrm{loc}}(TM_0 \oplus (T\mathcal{F})^0)^G$,
(3) $\Pi \circ [\tilde{\sigma}_1, \tilde{\sigma}_2]_{|M_0} = [\sigma_1, \sigma_2] \circ \pi$,

where the Courant brackets are taken in $\Gamma_{\mathrm{loc}}(TM \oplus T^*M)$ and $\Gamma_{\mathrm{loc}}(TM_G \oplus T^*M_G)$ as is appropriate. In other words, $[\tilde{\sigma}_1, \tilde{\sigma}_2]$ is a lift of $[\sigma_1, \sigma_2]$.

Proof. (1) Assume that $\tilde{\sigma}_1 = X + \xi$ and $\tilde{\sigma}_2 = Y + \eta$, where $X, Y \in \mathfrak{X}_{\text{loc}}(M)^G$ and $\xi, \eta \in \Omega^1_{\text{loc}}(M)^G$. Then

$$[\tilde{\sigma}_1, \tilde{\sigma}_2] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \frac{1}{2} d(\xi(Y) - \eta(X))$$

is clearly G-invariant.

(2) First $[X, Y]_{|M_0} \in \mathfrak{X}(M_0)$ since X and Y belong to $\mathfrak{X}(M_0)$. Now,

$$(\mathcal{L}_X \eta)(\hat{A})_{|M_0} = X\left(\eta(\hat{A})\right)_{|M_0} - \eta([X, \hat{A}])_{|M_0}, \quad \forall A \in \mathfrak{g}.$$

Since $\eta(\hat{A})_{|M_0} = 0$ and X is tangent to M_0, we have $X(\eta(\hat{A}))_{|M_0} = 0$. Moreover, $[X, \hat{A}] = 0$ because X is G-invariant. Hence $(\mathcal{L}_X \eta)(\hat{A})_{|M_0} = 0$. In other words, $\mathcal{L}_X \eta_{|M_0} \in (T\mathcal{F})^0$. Similarly, $\mathcal{L}_Y \xi_{|M_0} \in (T\mathcal{F})^0$. Finally, $d(\xi(Y) - \eta(X)) \in (T\mathcal{F})^0$. For $\xi(Y) - \eta(X)$ is a G-invariant function. Therefore we have $[\tilde{\sigma}_1, \tilde{\sigma}_2]_{|M_0} \in \Gamma_{\text{loc}}(TM_0 \oplus (T\mathcal{F})^0)^G$.

(3) By $\hat{\sigma}_1, \hat{\sigma}_2$, we denote the sections of $\Gamma_{\text{loc}}(TM_0 \oplus T^*M_0)$ obtained by considering the cotangent parts of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ as elements in T^*M_0. It is clear that $\tilde{\sigma}_i \overset{i}{\rightsquigarrow} \hat{\sigma}_i$ and $\hat{\sigma}_i \overset{\pi}{\rightsquigarrow} \sigma_i$. The conclusion follows from Lemma 2.2. $\quad\square$

Proposition 4.5. *Assume that, given any element $v_n \in TM_G \oplus T^*M_G$, there exists a local smooth section ζ of $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0) \to M_0$ around some point $m \in \pi^{-1}(n)$ such that $\zeta|_m$ is a lift of v_n. Then the following assertions hold:*

(1) *The subbundles $E'_\pm \subset T_\mathbb{C} M_G \oplus T_\mathbb{C}^* M_G$ are smooth.*

(2) *Any smooth (global) section of $\Gamma(E'_\pm)$ locally admits a lift to a local section in $\Gamma(E_\pm)^G$.*

Proof. (1) Since $E'_\pm$ has constant rank, it suffices to prove that any element $v_n \in E'_\pm$ admits a smooth local section passing through it. Let ζ be a local section of $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$ defined in an open neighborhood V containing m with $\pi(m) = n$ such that $\zeta|_m$ is a lift of v_n. Shrinking V if necessary, we can extend ζ to a local section of $TM \oplus T^*M \to M$, denoted by the same symbol ζ, and defined in an open neighborhood U of M_0 such that $U \cap M_0 = V$. Now take a slice L in U through the point m and transversal to the G-orbits. Such a slice always exists if U is taken small enough. Let $\tilde{\sigma}$ be the unique G-invariant section of $\Gamma((TM \oplus T^*M)_{|U})$ such that $\tilde{\sigma}_{|L} = \zeta_{|L}$. Set $\tilde{\tau} = \frac{1}{2}(I \mp \mathbf{i}J) \circ \tilde{\sigma}$. It is clear that $\tilde{\tau} \in \Gamma(E_{\pm|U})$ is a local G-invariant smooth section such that $\tilde{\tau}_{|U \cap M_0} \in \Gamma_{\text{loc}}(TM_0 \oplus (T\mathcal{F})^0)^G$. Therefore, it induces a smooth section $\tau \in \Gamma(E'_{\pm|\pi(U)})$ through the element v_n.

(2) Let $\nu \in \Gamma(E'_+)$ be any smooth section around the point $n \in M_G$. Choose a basis $\{v^i_n\}_{i=1,\ldots,k}$ of $E'_{+|n}$. Let τ^i be the local smooth section of E'_+ through the element v^i_n constructed as in (1). Then $\{\tau^i|_{n'}\}_{i=1,\ldots,k}$ is a basis of $E'_{+|n'}$ when n' is in a sufficiently small neighborhood V around n. Thus $\nu = \sum f_i \tau^i$ for some $f_i \in C^\infty(V)$. Let $\tilde{\tau}^i \in \Gamma(E_{\pm|U})$ be the lift of τ^i as in (1) defined on an open neighborhood U such that $U \cap M_0 = \pi^{-1}(V)$. Choose $\tilde{f}_i \in C^\infty(U)^G$ such that $\tilde{f}_{i|M_0 \cap U} = \pi^* f_i$. Such $\tilde{f}_i$ always exist if U is chosen sufficiently small. Then $\tilde{\nu} = \sum \tilde{f}_i \tilde{\tau}^i$ is a lift of ν in $\Gamma(E_\pm)^G$. $\quad\square$

We are now ready to state the main theorem of this paper.

Theorem 4.6. *Let M be a manifold endowed with an action of a Lie group G. Let J be a G-invariant generalized complex structure on M. And let M_0 be a G-invariant submanifold of M where the G-action is free and proper so that the quotient $M_G = M_0/G$ is a manifold. Assume that, given any element $v_n \in TM_G \oplus T^*M_G$, there exists a local smooth section ζ of $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0) \to M_0$ around some point $m \in \pi^{-1}(n)$ such that $\zeta|_m$ is a lift of v_n. Then there is an induced generalized complex structure J_G on M_G.*

Proof. This follows from Corollary 4.2, Proposition 4.5 and Lemma 4.4. $\quad\square$

Lemma 4.7. *The hypothesis of Proposition 4.5 is satisfied when*

$$(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0) \text{ is a generalized smooth subbundle} \tag{5}$$

and

$$J(T\mathcal{F} \oplus (TM_0)^0) \cap (TM_0 \oplus (T\mathcal{F})^0) \subset T\mathcal{F} \oplus (TM_0)^0. \tag{6}$$

Proof. For any vector $v_n \in TM_G \oplus T^*M_G$, according to Proposition 4.1 and Corollary 4.2, Eq. (6) implies the existence of a lift ζ_m of v_n in $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$. Since $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0) \to M_0$ is a generalized smooth subbundle, it admits a local section ζ around m in M_0 which extends ζ_m. $\quad\square$

Theorem 4.8. *Let M be a manifold endowed with an action of a Lie group G, and J a G-invariant generalized complex structure on M. Assume that M_0 is a G-invariant submanifold of M where the G-action is free and proper so that the quotient $M_G = M_0/G$ is a manifold. Then there is an induced generalized complex structure on M_G if the hypotheses* (5) *and* (6) *of Lemma* 4.7 *are satisfied.*

Theorem 4.8 has several important consequences.

Corollary 4.9. *Under the same hypothesis as in Theorem* 4.8, *if moreover*

$$J(T\mathcal{F} \oplus (TM_0)^0) = T\mathcal{F} \oplus (TM_0)^0, \tag{7}$$

then M_G admits an inherited generalized complex structure.

Proof. Since J is orthogonal and $TM_0 \oplus (T\mathcal{F})^0 = (T\mathcal{F} \oplus (TM_0)^0)^{\perp\langle\cdot,\cdot\rangle}$, Eq. (7) implies that $J(TM_0 \oplus (T\mathcal{F})^0) = TM_0 \oplus (T\mathcal{F})^0$. Hence $(TM_0 \oplus (T\mathcal{F})^0 \cap J(TM_0 \oplus (T\mathcal{F})^0)) = TM_0 \oplus (T\mathcal{F})^0$ is smooth. And Eq. (6) is satisfied since $T\mathcal{F} \oplus (TM_0)^0 \subset TM_0 \oplus (T\mathcal{F})^0$. The result follows from Theorem 4.8. $\quad\square$

Remark 4.10. This is exactly what happens in the case of Marsden–Weinstein reduction [23]. In this case, $J = \begin{pmatrix} 0 & -\omega_\flat^{-1} \\ \omega_\flat & 0 \end{pmatrix}$, and $M_0 = \mu^{-1}(0)$, where $\omega \in \Omega^2(M)$ is a symplectic form and $\mu : M \to \mathfrak{g}^*$ is an equivariant momentum map: $\hat{A}\lrcorner\omega = d\langle\mu, A\rangle, \forall A \in \mathfrak{g}$. One checks easily that $J(T\mathcal{F}) = (TM_0)^0$, and therefore $J(TM_0)^0 = T\mathcal{F}$ since $J^2 = -I$. Hence the conditions in Corollary 4.9 are satisfied.

More generally one sees that these conditions hold when J is a symplectic structure twisted by a B-field.

Corollary 4.11. *Let (M, ω) be a symplectic manifold endowed with an action of a Lie group G preserving the symplectic structure. Assume that there exists a G-invariant closed 2-form B on M and an equivariant momentum map $\mu : M \to \mathfrak{g}^*$.*

$$\hat{A}\lrcorner\omega = d\langle\mu, A\rangle, \quad \forall A \in \mathfrak{g}. \tag{8}$$

Assume that 0 is a regular value of μ and the G-action is free and proper. Thus $M_0 = \mu^{-1}(0)$ is a G-invariant submanifold of M and the quotient $M_G = M_0/G$ is a smooth manifold. If i^*B, where $i : M_0 \to M$ is the inclusion, is basic with respect to the G-action, then the generalized complex structure

$$J = \begin{pmatrix} 1 & 0 \\ B_\flat & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega_\flat^{-1} \\ \omega_\flat & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B_\flat & 1 \end{pmatrix}$$

on M induces a generalized complex structure on M_G.

Proof. Note that the assumption i^*B being basic implies that $TM_0 \oplus (T\mathcal{F})^0$ is stable under the bundle map $\begin{pmatrix} 1 & 0 \\ \pm B_\flat & 1 \end{pmatrix}$ of $TM \oplus TM^*$. By the discussion in Remark 4.10, we know that $TM_0 \oplus (T\mathcal{F})^0$ is also stable under $\begin{pmatrix} 0 & -\omega_\flat^{-1} \\ \omega_\flat & 0 \end{pmatrix}$. The conclusion thus follows from Corollary 4.9. $\quad\square$

Here is yet another important situation.

Corollary 4.12. *Under the same hypothesis as in Theorem* 4.8, *if moreover*

$$J(T\mathcal{F} \oplus (TM_0)^0) \cap (TM_0 \oplus (T\mathcal{F})^0) = 0, \tag{9}$$

then M_G admits an inherited generalized complex structure.

In particular, if $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$ is a usual complex structure, the above relation is nothing else than the Guillemin–Sternberg condition [11]:

$$TM = TM_0 \oplus j(T\mathcal{F}). \tag{10}$$

In this case, the induced generalized complex structure is still a complex structure.

Proof. Since Eq. (9) implies Eq. (6), Condition (5) of Proposition 4.1 holds. Together with Eq. (9), this implies that

$$J(TM_0 \oplus (T\mathcal{F})^0) \subset (TM_0 \oplus (T\mathcal{F})^0) \oplus J(T\mathcal{F} \oplus (TM_0)^0).$$

Since $T\mathcal{F} \oplus (TM_0)^0 \subset TM_0 \oplus (T\mathcal{F})^0$, the projection of $J(TM_0 \oplus (T\mathcal{F})^0)$ onto $TM_0 \oplus (T\mathcal{F})^0$ parallel to $J(T\mathcal{F} \oplus (TM_0)^0)$ is equal to $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$. Therefore, we have

$$J(TM_0 \oplus (T\mathcal{F})^0) = \Big(TM_0 \oplus (T\mathcal{F})^0\Big) \cap J\Big(TM_0 \oplus (T\mathcal{F})^0\Big) \oplus J(T\mathcal{F} \oplus (TM_0)^0),$$

which implies that $(TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$ must be a smooth subbundle. Thus the conclusion follows from Theorem 4.8.

The second part of Corollary 4.12 can be easily checked and is left for the reader. $\square$

Corollary 4.13. *Under the same hypothesis as in Theorem* 4.8, *if moreover there exists a Riemann metric g such that $(T\mathcal{F} \oplus (TM_0)^\perp)^\perp \oplus (T\mathcal{F} \oplus (TM_0)^\perp)^0$ is J-stable, then M_G admits an inherited generalized complex structure. Here $\perp$ refers to the orthogonal subspace in TM with respect to the metric g.*

In particular, if (M, g, j) is a G-invariant Hermitian manifold such that $T\mathcal{F} \oplus (TM_0)^\perp$ is j-stable, then M_G admits an inherited complex structure.

Proof. Since $L := \big(T\mathcal{F} \oplus (TM_0)^\perp\big)^\perp \oplus \big(T\mathcal{F} \oplus (TM_0)^\perp\big)^0$ is J-stable, one has $L \subset (TM_0 \oplus (T\mathcal{F})^0) \cap J(TM_0 \oplus (T\mathcal{F})^0)$. Let Q be the projection of $(TM \oplus T^*M)_{|M_0}$ onto $(T\mathcal{F})^\perp \oplus ((TM_0)^\perp)^0$ parallel to $T\mathcal{F} \oplus (TM_0)^0$. Then, since $T\mathcal{F} \oplus (TM_0)^0 \subset TM_0 \oplus (T\mathcal{F})^0$, one has

$$Q(TM_0 \oplus (T\mathcal{F})^0) \subset \Big(T\mathcal{F} \oplus (TM_0)^\perp\Big)^\perp \oplus \Big(T\mathcal{F} \oplus (TM_0)^\perp\Big)^0 = L.$$

Now fix an element v of $TM_G \oplus T^*M_G$, and let $\tilde{v} \in TM_0 \oplus (T\mathcal{F})^0$ be a lift of v. Then $Q(\tilde{v})$ is a lift of v in L. The bundle L being smooth, there exists a section s of L extending $Q(\tilde{v})$. Hence the hypothesis of Proposition 4.5 is satisfied and the result follows from Theorem 4.6.

In particular, if $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$, where j preserves $T\mathcal{F} \oplus (TM_0)^\perp$, and J is orthogonal with respect to g, then the smooth subbundle L of $TM_0 \oplus (T\mathcal{F})^0$ is indeed J-invariant. $\square$

Remark 4.14. Note that the induced complex structure on the quotient space in Kähler reduction can be considered as a special case of the above situation.

Corollary 4.15. *Let (M, j) be a complex manifold endowed with an action of a Lie group G such that j is G-invariant. Let M_0 be a G-invariant submanifold of M, where the G-action is free and proper so that the quotient $M_G = M_0/G$ is a manifold. Assume that, for any $X_n \in T_nM_G$, $\xi_n \in T_n^*M_G$, there exists a point $m \in M_0$ with $\pi(m) = n$ and local sections $\tilde{X} \in \Gamma_{\mathrm{loc}}(TM_0 \cap jTM_0)$, $\tilde{\xi} \in \Gamma_{\mathrm{loc}}((T\mathcal{F} + jT\mathcal{F})^0)$ such that $\pi_*(\tilde{X}|_m) = X_n$ and $\pi^*\xi_n = \tilde{\xi}|_m$. Then M_G admits an inherited generalized complex structure, which is still a complex structure.*
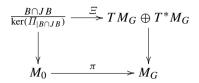
In particular, the above conditions are satisfied if $TM_0 \cap jTM_0$ and $T\mathcal{F} + jT\mathcal{F}$ are generalized smooth subbundles of TM_0.

Proof. The above hypothesis is the projection of the hypothesis of Proposition 4.5 onto the tangent and cotangent bundles. $\square$

5. Description of J_G

This section is devoted to an explicit description of the induced generalized complex structure J_G under the assumption that the hypothesis of Theorem 4.6 holds.

Let B denote the vector bundle $TM_0 \oplus (T\mathcal{F})^0$ and $B^\perp$ its $\langle \cdot, \cdot \rangle$-orthogonal subbundle: $T\mathcal{F} \oplus (TM_0)^0$. As in Corollary 4.2, we know that every vector in $TM_G \oplus T^*M_G$ has a lift in $B \cap JB$. In other words, the restriction of the bundle map Π (as in Eq. (3)) to $B \cap JB$ is surjective. Then $\Pi_{|B \cap JB}$ induces a G-invariant vector bundle map
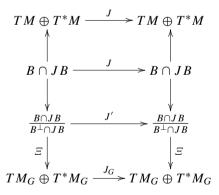
$$\frac{B \cap JB}{\ker(\Pi_{|B \cap JB})} \xrightarrow{\quad \Xi \quad} TM_G \oplus T^*M_G$$

$$M_0 \xrightarrow{\quad \pi \quad} M_G$$

which is indeed a pullback bundle. Note that $\ker_{|B \cap JB} = B^\perp \cap JB$. Now, by Corollary 4.2 and Proposition 4.1, we have $JB^\perp \cap B \subset B^\perp$. Moreover since $B^\perp \subset B$, we thus have $JB^\perp \cap B \subset B^\perp \cap JB$. Since $J^2 = -I$, it follows that $JB^\perp \cap B = B^\perp \cap JB$. It is simple to see that $B^\perp \cap JB$ is J-stable. Therefore, $J_{|B \cap JB}$ induces a bundle isomorphism

$$\frac{B \cap JB}{B^\perp \cap JB} \xrightarrow{\quad J' \quad} \frac{B \cap JB}{B^\perp \cap JB}$$

$$M_0 \xrightarrow{\quad id \quad} M_0$$

whose G-invariance implies that it factorizes through Ξ so that it induces a bundle map J_G as the inherited generalized complex structure

$$TM_G \oplus T^*M_G \xrightarrow{\quad J_G \quad} TM_G \oplus T^*M_G$$

$$M_G \xrightarrow{\quad id \quad} M_G.$$

Thus we obtain the following

Theorem 5.1. *Under the same hypothesis as for Theorem 4.6, the induced generalized complex structure J_G on M_G can be described by the following commutative diagram:*

$$
\begin{array}{ccc}
TM \oplus T^*M & \xrightarrow{\quad J \quad} & TM \oplus T^*M \\
\uparrow & & \uparrow \\
B \cap JB & \xrightarrow{\quad J \quad} & B \cap JB \\
\downarrow & & \downarrow \\
\frac{B \cap JB}{B^\perp \cap JB} & \xrightarrow{\quad J' \quad} & \frac{B \cap JB}{B^\perp \cap JB} \\
\Xi \downarrow & & \downarrow \Xi \\
TM_G \oplus T^*M_G & \xrightarrow{\quad J_G \quad} & TM_G \oplus T^*M_G
\end{array}
$$

Hence, if $\tilde{v} \in B \cap JB$ is a lift of $v \in TM_G \oplus T^*M_G$, then $J\tilde{v} \in B \cap JB$ is a lift of $J_G v \in TM_G \oplus T^*M_G$.

Remark 5.2. In [9], Crainic showed that a generalized complex structure J is an endomorphism of the generalized tangent bundle $TM \oplus T^*M$ built out of a Poisson bivector π, an endomorphism N of the tangent bundle and a 2-form σ:

$$J = \begin{pmatrix} N & \pi^\sharp \\ \sigma_\flat & -N^* \end{pmatrix}$$

with $N^2 + \pi^\sharp \sigma_\flat = -\mathrm{idn}$, which satisfy some compatibility conditions resembling those given by Kosmann-Schwarzbach, Magri and Morosi [17,22] in their definition of Poisson Nijenhuis structures (see also [18,28]). It would be interesting to investigate the meaning of each component for the reduction J_G.

6. Reduction of generalized Kähler structures

The *generalized Kähler structure* as introduced by Hitchin [13] and Gualtieri [10] consists of a pair of commuting generalized complex structures (J_1, J_2) such that $\langle J_1 J_2 \cdot, \cdot \rangle$ is positive definite.

Example 6.1. If (M, ω, j) is a usual Kähler manifold, then

$$J_1 = \begin{pmatrix} 0 & -\omega_\flat^{-1} \\ \omega_\flat & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$$

defines a generalized Kähler structure.

As in Section 4, we assume that M_0 is a G-invariant submanifold of M where the G-action is free and proper so that the quotient $M_G = M_0/G$ is a manifold. Since $TM \oplus T^*M$ decomposes as the direct sum of four different subbundles with each being constituted of the common eigenvectors of J_1 and J_2 determined by a given pair of eigenvalues, it is natural to ask whether the reduction procedure outlined in Section 4 can be applied to J_1 and J_2 simultaneously to get a generalized Kähler structure on the quotient M_G.

Let us introduce some notation. By $E^\pm$ and $E_\pm$ we denote the $\pm i$-eigenbundles of J_1 and J_2 respectively. Let $B_\mathbb{C} = B \otimes \mathbb{C}$. For any subbundle V of $T_\mathbb{C} M \oplus T^*_\mathbb{C} M$, $\Phi(V)$ denotes the subbundle $\Pi(V \cap B_\mathbb{C})$ of $T_\mathbb{C} M_G \oplus T^*_\mathbb{C} M_G$, where Π is the bundle map as in Eq. (3).

Proposition 6.2. *Assume that (J_1, J_2) is a G-invariant Kähler structure.*

(1) *The following two statements are equivalent:*

$$T_\mathbb{C} M_G \oplus T^*_\mathbb{C} M_G = \Phi(E^+ \cap E_+) + \Phi(E^+ \cap E_-) + \Phi(E^- \cap E_+) + \Phi(E^- \cap E_-) \tag{11}$$

and

$$\Pi(B \cap J_1 B \cap J_2 B) = T M_G \oplus T^* M_G,$$

where, as in Section 5, $B = TM_0 \oplus (T\mathcal{F})^0$.

(2) *If the above condition is satisfied, then the above sum is direct and*

$$\Phi(E^\pm \cap E_\pm) = \Phi(E^\pm) \cap \Phi(E_\pm).$$

Proof. (1) $\Rightarrow$ Take $v \in T M_G \oplus T^* M_G$. By assumption, it can be decomposed as
$$v = \Pi(\tilde{v}_+^+) + \Pi(\tilde{v}_-^+) + \Pi(\tilde{v}_+^-) + \Pi(\tilde{v}_-^-),$$
where $\tilde{v}_\pm^\pm \in E^\pm \cap E_\pm \cap B_\mathbb{C}$. Let
$$\tilde{v} = \tilde{v}_+^+ + \tilde{v}_-^+ + \tilde{v}_+^- + \tilde{v}_-^-.$$
Then $\tilde{v} \in B_\mathbb{C}$ and $\Pi(\tilde{v}) = v$. Hence $\tilde{v}$ is a lift of v. Moreover,
$$J_1 \tilde{v} = i(\tilde{v}_+^+ + \tilde{v}_-^+ - \tilde{v}_+^- - \tilde{v}_-^-) \in B_\mathbb{C} \quad \text{and} \quad J_2 \tilde{v} = i(\tilde{v}_+^+ - \tilde{v}_-^+ + \tilde{v}_+^- - \tilde{v}_-^-) \in B_\mathbb{C}.$$
Hence, $\tilde{v} \in J_1 B_\mathbb{C}$ and $\tilde{v} \in J_2 B_\mathbb{C}$ since $J_1^2 = J_2^2 = -I$. Therefore $\text{Re}(\tilde{v}) \in B \cap J_1 B \cap J_2 B$ is a lift of $v \in T M_G \oplus T^* M_G$.

$\Leftarrow$ Conversely, if any $v \in T M_G \oplus T^* M_G$ has a lift $\tilde{v}$ in $B \cap J_1 B \cap J_2 B$, one has
$$v = \Pi \left(\frac{1}{2}(I - iJ_1)\frac{1}{2}(I - iJ_2)\tilde{v} \right) + \left(\frac{1}{2}(I - iJ_1)\frac{1}{2}(I + iJ_2)\tilde{v} \right)$$
$$+ \left(\frac{1}{2}(I + iJ_1)\frac{1}{2}(I - iJ_2)\tilde{v} \right) + \left(\frac{1}{2}(I + iJ_1)\frac{1}{2}(I + iJ_2)\tilde{v} \right).$$
Eq. (11) thus follows.

(2) Finally, since

$$\Phi(E^\pm \cap E_\pm) \subseteq \Phi(E^\pm) \cap \Phi(E_\pm), \tag{12}$$

the condition

$$T_\mathbb{C} M_G \oplus T^*_\mathbb{C} M_G = \Phi(E^+ \cap E_+) + \Phi(E^+ \cap E_-) + \Phi(E^- \cap E_+) + \Phi(E^- \cap E_-) \tag{13}$$

implies that

$$T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G = \Phi(E^+) \cap \Phi(E_+) + \Phi(E^+) \cap \Phi(E_-) + \Phi(E^-) \cap \Phi(E_+) + \Phi(E^-) \cap \Phi(E_-) \quad (14)$$

and

$$T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G = \Phi(E^+) + \Phi(E^-), \qquad T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G = \Phi(E_+) + \Phi(E_-).$$

Applying Proposition 4.1 to J_1 and J_2, we obtain

$$\Phi(E^+) \cap \Phi(E^-) = 0 \quad \text{and} \quad \Phi(E_+) \cap \Phi(E_-) = 0.$$

This implies that the sums in (13) and (14) must be direct. Hence Eq. (12) implies that $\Phi(E^\pm \cap E_\pm) = \Phi(E^\pm) \cap \Phi(E_\pm)$. $\quad\square$

Theorem 6.3. *Let* (M, J_1, J_2) *be a G-invariant generalized Kähler manifold. Assume that M_0 is a G-invariant submanifold of M where the G-action is free and proper so that the quotient $M_G = M_0/G$ is a manifold. If, for any $v_n \in TM_G \oplus T^*M_G$, there exists a point $m \in M_0$ and a local smooth section ζ of $B \cap J_1 B \cap J_2 B \to M_0$ around m such that $\Pi(\zeta|_m) = v_n$, then there exists an inherited generalized Kähler structure (J_{1G}, J_{2G}) on M_G such that*

$$\begin{aligned} \Pi(J_1 w) &= J_{1G} \Pi(w) \\ \Pi(J_2 w) &= J_{2G} \Pi(w) \end{aligned} \quad \forall w \in B \cap J_1 B \cap J_2 B. \quad (15)$$

In particular, this condition holds if

- $B = B \cap J_1 B \cap J_2 B + B^\perp$ *and*
- $B \cap J_1 B \cap J_2 B$ *is a generalized smooth subbundle.*

Proof. According to Theorem 4.6, the generalized complex structures J_1 and J_2 induce a pair of generalized complex structures J_{1G} and J_{2G} on M_G satisfying Eq. (15). Let $E'^{\pm}$ and $E'_{\pm}$ denote their $\pm\mathbf{i}$-eigenbundles respectively. By Proposition 6.2, we have

$$T_{\mathbb{C}}M_G \oplus T_{\mathbb{C}}^*M_G = \left(E'^{+} \cap E'_{+} \right) \oplus \left(E'^{+} \cap E'_{-} \right) \oplus \left(E'^{-} \cap E'_{+} \right) \oplus \left(E'^{-} \cap E'_{-} \right),$$

which implies that J_{1G} and J_{2G} commute.

Finally, for $v \in TM_G \oplus T^*M_G$, choose $w \in B \cap J_1 B \cap J_2 B$. By Eq. (15), we have

$$\langle J_{1G} J_{2G} v, v \rangle = \langle J_{1G} J_{2G} \Pi(w), \Pi(w) \rangle = \langle \Pi(J_1 J_2 w), \Pi(w) \rangle = \langle J_1 J_2 w, w \rangle,$$

where the last equality follows from the fact that $\Pi|_B$ preserves the scalar products. It thus follows that $\langle J_{1G} J_{2G} \cdot, \cdot \rangle$ is positive definite. This concludes the proof. $\quad\square$

The following corollary is to Theorem 6.3 what Corollary 4.13 is to Theorem 4.6.

Corollary 6.4. *Let* (M, J_1, J_2) *be a G-invariant generalized Kähler manifold. Assume that M_0 is a G-invariant submanifold of M, where the G-action is free and proper so that the quotient $M_G = M_0/G$ is a manifold. Assume that there exists a Riemann metric g such that $(T\mathcal{F} \oplus (TM_0)^\perp)^\perp \oplus (T\mathcal{F} \oplus (TM_0)^\perp)^0$ is stable under both J_1 and J_2; then M_G admits an inherited generalized complex structure.*

Proof. The proof is exactly the same as for Corollary 4.13 except that, here, since L is both J_1- and J_2-invariant, one has $L \subset B \cap J_1 B \cap J_2 B$. Then the conclusion follows from Theorem 6.3. $\quad\square$

As proved by Gualtieri [10], a generalized Kähler structure (J_1, J_2) on a manifold M is fully characterized by a quadruple (g, b, J_+, J_-), where g is a Riemannian metric, b is a 2-form and $J_\pm$ are two (integrable) complex structures on M, compatible with g, such that

$$\mathrm{d}b(X, Y, Z) = \mathrm{d}\omega_\pm(J_\pm X, J_\pm Y, J_\pm Z), \quad \forall X, Y, Z \in \mathfrak{X}(M)$$

with $\omega_\pm \in \Omega^2(M)$ defined by $\langle \omega_\pm X, Y \rangle = \pm g(J_\pm X, Y)$. These data explicitly determine the pair of generalized complex structures

$$J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ + J_- & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & -(J_+^* + J_-^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

$$J_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ - J_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(J_+^* - J_-^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}.$$

Here, for simplicity, we identify a 2-form with its associated bundle map.

Using exactly this same g and applying Corollary 6.4, we are led to the main result of this section, which generalizes the usual Kähler reduction [11,12,16].

Theorem 6.5. *Let* (M, J_1, J_2) *be a G-invariant generalized Kähler manifold. Assume that* M_0 *is a G-invariant submanifold of* M, *where the G-action is free and proper so that the quotient* $M_G = M_0/G$ *is a manifold. Let* $(g, b, \omega_+, \omega_-)$ *be the quadruple associated with a generalized Kähler structure* (J_1, J_2) *as in* [10]. *If* $\omega_\pm(T\mathcal{F}) = (TM_0)^0$ *and* $(T\mathcal{F})^\perp \cap TM_0$ *is stable under* $\omega_\pm^{-1}b$, *then* M_G *inherits a generalized Kähler structure.*

Proof. It is sufficient to check that the conditions in Corollary 6.4 are satisfied.

First, since $\omega_\pm$ are antisymmetric and non-degenerate, $\omega_\pm(T\mathcal{F}) = (TM_0)^0$ implies $\omega_\pm(TM_0) = (T\mathcal{F})^0$.

Second, $\omega_\pm(T\mathcal{F}) = (TM_0)^0$ also implies $\omega_\pm(T\mathcal{F}^\perp) = (TM_0^\perp)^0$ because $J_\pm$ are isometric with respect to g. Therefore,

$$\omega_\pm\left(T\mathcal{F} \oplus (TM_0)^\perp\right)^\perp = \omega_\pm(T\mathcal{F}^\perp \cap TM_0) = \left(TM_0^\perp\right)^0 \cap (T\mathcal{F})^0 = \left(T\mathcal{F} \oplus (TM_0)^\perp\right)^0. \tag{16}$$

Finally, one easily checks that the image of any

$$v + f \in \left(T\mathcal{F} \oplus (TM_0)^\perp\right)^\perp \oplus \left(T\mathcal{F} \oplus (TM_0)^\perp\right)^0$$

under

$$J_1 \pm J_2 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_\pm & -\omega_\pm^{-1} \\ \omega_\pm & -J_\pm^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

lies in $\left(T\mathcal{F} \oplus (TM_0)^\perp\right)^\perp \oplus \left(T\mathcal{F} \oplus (TM_0)^\perp\right)^0$ if and only if both Eq. (16) and the condition $\omega_\pm^{-1}b((T\mathcal{F})^\perp \cap TM_0) \subset (T\mathcal{F})^\perp \cap TM_0$ hold. $\square$

Note that $\omega_\pm(T\mathcal{F}) = (TM_0)^0$ is satisfied when M_0 is the zero level set of an equivariant momentum map. Thus the theorem above reduces to the usual Kähler reduction when M is a Kähler manifold [11,12,16].

Acknowledgments

References

[1] J. Barton, M. Stiénon, Generalized complex submanifolds. arXiv:math.DG/0603480.
[2] O. Ben-Bassat, M. Boyarchenko, Submanifolds of generalized complex manifolds, J. Symplectic Geom. 2 (3) (2004) 309–355.
[3] G. Blankenstein, Implicit Hamiltonian systems: Symmetry and interconnection, Ph.D. Thesis, University of Twente, The Netherlands, November 2000.
[4] G. Blankenstein, A.J. van der Schaft, Symmetry and reduction in implicit generalized Hamiltonian systems, Rep. Math. Phys. 47 (2001) 57–100.
[5] G. Blankenstein, T.S. Ratiu, Singular reduction of implicit Hamiltonian systems, Rep. Math. Phys. 53 (2004) 211–260.
[6] H. Bursztyn, G.R. Cavalcanti, M. Gualtieri, Reduction of Courant algebroids and generalized complex structures, Adv. Math. 211 (2) (2007) 726–765.
[7] H. Bursztyn, O. Radko, Gauge equivalence of Dirac structures and symplectic groupoids, Ann. Inst. Fourier (Grenoble) 53 (2003) 309–337.
[8] T. Courant, Dirac manifolds, Trans. Amer. Math. Soc. 319 (1990) 631–661.
[9] M. Crainic, Generalized complex structures and Lie brackets. arXiv:math.DG/0412097.
[10] M. Gualtieri, Generalized complex geometry. arXiv:math.DG/0401221.
[11] V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (1982) 515–538.
[12] N.J. Hitchin, A. Karlhede, U. Lindström, M. Roček, Hyper–Kähler metrics and supersymmetry, Comm. Math. Phys. 108 (1987) 535–589.
[13] N. Hitchin, Generalized Calabi–Yau manifolds, Q. J. Math. 54 (2003) 281–308.

[14] S. Hu, Hamiltonian symmetries and reduction in generalized geometry. arXiv:math.DG/0509060.
[15] J. Huebschmann, Kähler spaces, nilpotent orbits, and singular reduction, Mem. Amer. Math. Soc. 172 (2004).
[16] F. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, in: Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984.
[17] Y. Kosmann-Schwarzbach, F. Magri, Poisson–Nijenhuis structures, Ann. Inst. H. Poincaré Phys. Théor. 53 (1) (1990) 35–81.
[18] U. Lindström, R. Minasian, A. Tomasiello, M. Zabzine, Generalized complex manifolds and supersymmetry, Comm. Math. Phys. 257 (2005) 235–256.
[19] Z.-J. Liu, A. Weinstein, P. Xu, Manin triples for Lie bialgebroids, J. Differential Geom. 45 (1997) 547–574.
[20] K. Mackenzie, P. Xu, Lie bialgebroids and Poisson groupoids, Duke Math. J. 73 (1994) 415–452.
[21] Y. Lin, S. Tolman, Symmetries in generalized Kähler geometry, Comm. Math. Phys. 268 (1) (2006) 199–222.
[22] F. Magri, C. Morosi, On the reduction theory of the Nijenhuis operators and its applications to Gelfand–Dikii equations, in: Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics, Vol. II (Torino, 1982), vol. 117, 1983, pp. 599–626.
[23] J. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1974) 121–130.
[24] J.E. Marsden, T.S. Ratiu, Reduction of Poisson manifolds, Lett. Math. Phys. 11 (1984) 161–170.
[25] J.-P. Ortega, T.S. Ratiu, Singular reduction of Poisson manifolds, Lett. Math. Phys. 46 (1998) 359–372.
[26] J.-P. Ortega, T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, in: Progress in Mathematics, vol. 222, Birkhäuser-Verlag, Boston, 2004.
[27] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math. (2) 134 (1991) 375–422.
[28] M. Stiénon, P. Xu, Poisson quasi-Nijenhuis manifolds, Comm. Math. Phys. 270 (3) (2007) 709–725.
[29] I. Vaisman, Reduction and submanifolds of generalized complex manifolds, Differential Geom. Appl. 25 (2) (2007) 147–166.